# ON THE STABILITY OF A FREE GYROSTAT

## (K USTOICHIVOSTI SVOBODNOGO GIROSTATA)

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The methods of solving problems of stability of motion of a liquid-filled gyrostat with one point fixed [1], can be extended to free gyrostats moving in a Newtonian force field [2].

We consider here the problem of a gyrostat consisting of the rigid body $T_1$ and the rotors $T_2$, whose axes are fixed in $T_1$. The friction in the rotor axes and other dissipative effects are neglected. The general theory and Liapunov's criterion of stability of motion for gyrostats with one point fixed have been thoroughly investigated in the past, for example in [1]. In our work we have obtained sufficient conditions of stability for one particular solution of the equations of motion of a free gyrostat in the Newtonian force field.

**1.** Let 0 be the origin of a fixed Cartesian coordinate system, $\xi$, $\eta$ and $\zeta$ coinciding with the center of attraction. The gyrostat moves in a Newtonian central gravitational field, and the axes of the moving coordinate system $x$, $y$ and $z$ coincide with the principal central axes of inertia of the gyrostat.

Let $A$, $B$ and $C$ be the principal central moments of inertia of the gyrostat and let $M$ be its mass.

Our mechanical system, consisting of the solid casing $T_1$ and the symmetric rotors $T_2$, will be denoted by the single letter $T$.

The angular momentum of the system $T$ with respect to 0 equals

$$\mathbf{K}_0 = \mathbf{R} \times M\mathbf{V} + \mathbf{K} \qquad \left(R^2 = \xi^2 + \eta^2 + \zeta^2,\ V^2 = \left(\frac{d\xi}{dt}\right)^2 + \left(\frac{d\eta}{dt}\right)^2 + \left(\frac{d\zeta}{dt}\right)^2\right)$$

Here $\mathbf{R}$ is the radius vector of the center of mass of the whole system,

$\mathbf{V}$ is its velocity and $\mathbf{K}$ is the angular momentum of the gyrostat with respect to the König axes (axes through the center of gravity). From the conditions of the problem it is clearly seen that the theorem of König on the angular momentum is here applicable. Thus

$$\mathbf{K} = \mathbf{K}_T + \mathbf{K}_2$$

where $\mathbf{K}_T$ is the angular momentum of the whole system considered as a single body, and $\mathbf{K}_2$ is the angular momentum of the relative motions of $T_2$.

If the components of the vector of instantaneous angular velocity $\boldsymbol{\omega}$ of the body $T_1$ along the moving axes are $p$, $q$ and $r$, then the components of the vector $\mathbf{K}_T$ along these axes will be

$$Ap,\ Bq,\ Cr$$

Let the $x$, $y$ and $z$ projections of $\mathbf{K}_2$ be $k_1$, $k_2$ and $k_3$, respectively.

| | $x$ | $y$ | $z$ |
|---|---|---|---|
| $\xi$ | $\alpha_1$ | $\alpha_2$ | $\alpha_3$ |
| $\eta$ | $\beta_1$ | $\beta_2$ | $\beta_3$ |
| $\zeta$ | $\gamma_1$ | $\gamma_2$ | $\gamma_3$ |

Let $\tau_1$, $\tau_2$ and $\tau_3$ be the direction cosines of $R$ in the system $x$, $y$ and $z$, and let the direction cosines between the axes $\xi$, $\eta$ and $\zeta$ and $x$, $y$ and $z$ be given by the table on the right.

By König's theorem the equations of motion of the system are

$$M\frac{d^2\xi}{dt^2} = \frac{\partial U}{\partial \xi} \qquad (\xi\ \eta\ \zeta) \tag{1.1}$$

$$A\frac{dp}{dt} + \frac{dk_1}{dt} + (C - B)\,qr + qk_3 - rk_2 = L_x \qquad (p\ q\ r,\ 1, 2, 3,\ ABC,\ xyz) \tag{1.2}$$

Here the symbols in parentheses indicate that the remaining two equations in (1.1) and (1.2) are obtained by cyclic permutation of the indicated letters; $L_x$, $L_y$ and $L_z$ are the moments of the Newtonian forces acting on the system $T$ about the axes indicated by subscripts.

In real mechanical systems the ratio of a characteristic dimension to $R$ is of the order of $10^{-4}$ to $10^{-6}$; consequently, the Newtonian potential or the force function $U$ can be written as [3]

$$U = \frac{\mu M}{R} - \frac{3\mu}{2R^3}\left(A\tau_1^2 + B\tau_2^2 + C\tau_3^2 - \frac{A + B + C}{3}\right) \tag{1.3}$$

Now

$$L_x = \frac{3\mu}{R^3}(C - B)\,\tau_3\tau_2, \qquad L_y = \frac{3\mu}{R^3}(A - C)\,\tau_1\tau_3, \qquad L_z = \frac{3\mu}{R^3}(B - A)\,\tau_2\tau_1 \tag{1.4}$$

It is easy to see that

$$\begin{aligned}\tau_1 &= \alpha_1 \frac{\xi}{R} + \beta_1 \frac{\eta}{R} + \gamma_1 \frac{\zeta}{R}\\ \tau_2 &= \alpha_2 \frac{\xi}{R} + \beta_2 \frac{\eta}{R} + \gamma_2 \frac{\zeta}{R}\\ \tau_3 &= \alpha_2 \frac{\xi}{R} + \beta_3 \frac{\eta}{R} + \gamma_3 \frac{\zeta}{R}\end{aligned} \qquad (1.5)$$

and $U$ depends on all the direction cosines $\alpha_i$, $\beta_i$, $\gamma_i$ ($i = 1, 2, 3$). The components $L_x$, $L_y$ and $L_z$ of the vector of the gravitational moment acting on the system $T$ are expressed either as functions of $\tau_1$, $\tau_2$ and $\tau_3$, or of $\alpha_i$, $\beta_i$, $\gamma_i$ ($i = 1, 2, 3$) by the known formulas [4], or directly by using (1.4) or (1.5). The system of equations (1.1) and (1.2) should be supplemented by the kinematic equations of Poisson for the direction cosines and by the equations of the relative motion of the bodies $T_2$. In our case the equations of the relative motion should have the form of equations of motion of a rigid body with a fixed axis.

**2.** We shall investigate the special case of the motion of a free symmetric gyrostat

$$A = C, \qquad k_1 = k_3 = 0, \qquad k_2 = k(t) \qquad (2.1)$$

where $k(t)$ is a bounded continuous function of time.

With the above, formula (1.3) becomes

$$U = \frac{\mu M}{R} - \frac{3\mu}{2R^3}\left[(B - A)\,\tau_2^2 - \frac{(B - A)}{3}\right] \qquad (2.2)$$

and the equations of motion become

$$M\frac{d^2\xi}{dt^2} = \frac{\partial U}{\partial \xi}, \qquad M\frac{d^2\eta}{dt^2} = \frac{\partial U}{\partial \eta}, \qquad M\frac{d^2\zeta}{dt^2} = \frac{\partial U}{\partial \zeta} \qquad (2.3)$$

$$\begin{aligned}&A\frac{dp}{dt} + (A - B)\,qr - rk(t) = \frac{3\mu}{R^3}(A - B)\,\tau_3\tau_2\\ &B\frac{dq}{dt} + \frac{dk(t)}{dt} = 0\\ &A\frac{dr}{dt} + (B - A)\,pq + pk(t) = \frac{3\mu}{R^3}(B - A)\,\tau_2\tau_1\end{aligned} \qquad (2.4)$$

Here $k(t)$ is assumed to be a known function of time and hence our system can be regarded as closed if we add to it also the equation of Poisson for the direction cosines. The equations of motion of a free gyrostat of the considered type in a Newtonian force field with the potential (2.2) and under conditions (2.1) can yield several first integrals.

From the second equation (2.4) the following integral is obtained

$$Bq + k(t) = H = \text{const} \tag{2.5}$$

Let us multiply equations (2.3) in turn by $d\xi/dt$, $d\eta/dt$ and $d\zeta/dt$ and equations (2.4) by $p$, $q$ and $r$; then let us add them and integrate the sum. Taking into account (1.5) and the kinematic equations of Poisson for $\alpha_i$, $\beta_i$, $\gamma_i$ ($i$ = 1, 2, 3), we obtain the energy integral (*vis vivae*)

$$M\left[\left(\frac{d\xi}{dt}\right)^2 + \left(\frac{d\eta}{dt}\right)^2 + \left(\frac{d\zeta}{dt}\right)^2\right] + A(p^2 + r^2) - 2U = \text{const} \tag{2.6}$$

where $U$ is given by (2.2)

Another first integral can be easily obtained by projecting the vector of angular momentum of $T$ with respect to 0 on the $\zeta$-axis, which, without any loss of generality, can be assumed to be perpendicular to the plane of the orbit. This first integral is

$$M\left(\xi\frac{d\eta}{dt} - \eta\frac{d\xi}{dt}\right) + Ap\gamma_1 + (Bq + k(t)\,\gamma_2 + Ar\gamma_3 = \text{const} \tag{2.7}$$

Further, we have the trivial relationships

$$\tau_1^2 + \tau_2^2 + \tau_3^2 = 1, \qquad \gamma_1^2 + \gamma_2^2 + \gamma_3^2 = 1 \tag{2.8}$$

By introducing spherical coordinates whose center coincides with the center of mass of the system and by taking into account (2.5), the integrals (2.6) and (2.7) will take the following form:

$$M\left[\left(\frac{dR}{dt}\right)^2 + R^2\left(\frac{d\psi}{dt}\right)^2 + R^2\cos^2\psi\left(\frac{d\varphi}{dt}\right)^2\right] + A(p^2 + \gamma^2) - 2U = \text{const}$$

$$MR^2\cos^2\psi\,\frac{d\varphi}{dt} + A(p\gamma_1 + q\gamma_3) + H\gamma_2 = \text{const} \tag{2.9}$$

3. The equations of motion (2.3) and (2.4) together with the kinematic equations of Poisson admit the following particular solution:

$$\begin{gathered} p = r = 0, \qquad q = B^{-1}(H - k(t)) \\ \gamma_1 = \gamma_3 = 0, \qquad \gamma_2 = 1; \qquad R = R_0, \qquad dR/dt = 0 \\ \psi = 0, \qquad d\psi/dt = 0, \qquad \varphi = \omega t + \varphi_0, \qquad d\varphi/dt = \omega = \text{const} \\ \tau_2 = 0, \qquad \tau_1 = \sin\Omega(t) \qquad \tau_3 = \cos\Omega(t), \qquad d\Omega/dt = \omega - q(t) \end{gathered} \tag{3.1}$$

The motion of a gyrostat corresponding to the above solution consists of the motion of the center of mass on the circular orbit with radius $R_0$ with constant angular velocity $\omega$, and the rotation of the gyrostat about its axis of symmetry (remaining perpendicular to the plane of the orbit) with angular velocity $q$, while the rotor performs the prescribed motion, such that $Bq + k(t) = H = \text{const}$. Our problem consists of investigating the Liapunov stability of the described non-perturbed motion with respect to the group of variables

$$p,\ r,\ H,\ \tau_2,\ \gamma_1,\ \gamma_2,\ \gamma_3,\ R,\ dR/dt,\ \psi,\ d\psi/dt,\ d\varphi/dt \tag{3.2}$$

**4.** Let

$$p,\ r,\ H+x,\quad \tau_2,\ \gamma_1,\ \gamma_3,\ \gamma_2 = 1 + x_2,\quad R = R_0 + x_3$$

$$\frac{dR}{dt} = \frac{dx_3}{dt} = \dot{x}_3,\qquad \psi,\ \frac{d\psi}{dt} = \dot{\psi},\qquad \frac{d\varphi}{dt} = \dot{\varphi} = \omega + \frac{dx_4}{dt} = \omega + \dot{x}_4$$

denote the variables which we shall investigate in the perturbed motion. The equations of motion of a one rotor gyrostat in a Newtonian force field under conditions (2.1) permit the integrals

$$V_1 = MR_0^2\dot{\psi}^2 + M\dot{x}_3^2 - MR_0^2\omega^2\psi^2 + \left(2MR_0\omega^2 + \right.$$
$$\left. + \frac{2\mu M}{R_0^2} + \frac{6(B-A)\mu}{R_0^4}\right)x_3 + \left(M\omega^2 - \frac{2\mu M}{R_0^3} - \frac{6(B-A)\mu}{R_0^5}\right)x_3^2 +$$
$$+ 4MR_0\omega\, x_3\dot{x}_4 + MR_0^2\dot{x}_4^2 + 2MR_0\dot{x}_4 + \frac{3\mu(B-A)}{R_0^3}\tau_2^2 + A(p^2+r^2) + o(3) = \text{const}$$

$$V_2 = MR_0^2\dot{x}_4 + 2MR_0\omega\, x_3 + 2MR_0x_3\dot{x}_4 + M\omega\, x_3^2 - MR_0^2\omega\psi^2 + A(p\gamma_1 + r\gamma_3) +$$
$$+ Hx_2 + x_1 + x_1x_2 + o(3) = \text{const}$$

$$V_3 = x_1 = \text{const},\quad V_4 = \gamma_1^2 + \gamma_3^2 + x_2^2 + 2x_2$$

Here $o(3)$ denotes all terms of the third order and higher with respect to the perturbations. The stability of the considered non-perturbed motion will be investigated by the direct method of Liapunov. The investigation is performed by examining the function of the variables (3.2) constructed by Chetaev's method [5] which is in the form of the combined first integrals of the equations of motion

$$W = V_1 - 2\omega(V_2 - V_3) + H\omega V_4 + \lambda_1V_2^2 + \lambda_2V_3^2 = MR_0^2\dot{\psi}^2 + M\dot{x}_3^2 + MR_0^2\omega^2\psi^2 +$$
$$+ (MR_0^2 + \lambda_1M^2R_0^4)\dot{x}_4^2 + 4\lambda_1M^2R_0^3\omega x_3\dot{x}_4 + \left(4\lambda_1M^2R_0^2\omega^2 - 3M\omega^2 - \frac{3(B-A)\mu}{R_0^2}\right)x_3^2 +$$
$$+ \frac{3(B-A)\mu}{R_0^3}\tau_2^2 + Ap^2 - 2\omega Ap\gamma_1 + H\omega\gamma_1^2 + Ar^2 - 2\omega Ar\gamma_3 +$$
$$+ H\omega\gamma_3^2 + (-2\omega + 2H\lambda_1)x_1x_2 + (H\omega + \lambda_1H^2)x_2^2 + \lambda_1x_1^2 + \lambda_2\dot{x}_1^2 +$$
$$+ 2\lambda_1MR_0^2H\, x_2\dot{x}_4 + 2\lambda_1MR_0^2x_1\dot{x}_4 + 4\lambda_1MR_0H\omega x_2x_3 + 4\lambda_1MR_0\omega\, x_1x_3 \tag{4.1}$$

In this quadratic form $\lambda_1$ and $\lambda_2$ are constants. According to Sylvester's criterion the necessary and sufficient condition for the quadratic form $W$ to be positive-definite is

$$B > A, \qquad H > A\omega \tag{4.2}$$

and also the positiveness of all principal diagonal minors of the determinant of the quadratic form

$$\| c_{ij} \| \quad (c_{ij} = c_{ji}) \quad (i, j = 1, 2, 3, 4)$$

$$c_{11} = \lambda_1 M R_0^2 \omega^2 - 3M\omega^2 - \frac{3(B-A)\mu}{R_0^5}; \qquad c_{12} = 2\lambda_1 M^2 R_0^3 \omega$$

$$c_{13} = 2\lambda_1 M R_0 H\omega, \quad c_{14} = 2\lambda_1 M R_0 \omega, \quad c_{22} = MR_0^2 + \lambda_1 M^2 R_0^4$$

$$c_{23} = \lambda_1 M R_0^2 H, \qquad C_{24} = \lambda_1 M R_0^2, \qquad c_{33} = H\omega + \lambda_1 H^2$$

$$c_{34} = -\omega + \lambda_1 H, \qquad c_{44} = \lambda_1 + \lambda_2$$

The last requirement can be satisfied by suitable selection of the constants $\lambda_1$ and $\lambda_2$ under the condition $H < MR_0^2\omega/3$, which in practical cases is usually satisfied.

Consequently, when the condition (4.2) is satisfied and the constants $\lambda_1$ and $\lambda_2$ are suitably selected, the quadratic form (4.1) will be positive-definite with respect to all the variables, and it can serve in our case as the Liapunov function, since $dW/dt = 0$ on the strength of the equations of the perturbed motion.

By Liapunov's theorem, then, the non-perturbed motion of a gyrostat with one rotor, whose angular momentum satisfies the condition

$$Bq + k(t) - A\omega > 0$$

is stable.

The inequality obtained shows that a gyrostat of the considered type rotates in its orbit as if it were a single body with its principal moment of inertia changed to $H/\omega$, satisfying also the inequality $H/\omega > A$. When $k(t) = 0$ and $q = \omega$ we are left with the only condition $B > A$.

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